

Searching for pairing energies in phase space

M. Calixto¹, O. Castaños² and E. Romera³

¹ *Departamento de Matemática Aplicada, Universidad de Granada, Fuentenueva s/n, 18071 Granada, Spain*

² *Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México, Apdo. Postal 70-543 México 04510 D.F. and*

³ *Departamento de Física Atómica, Molecular y Nuclear and Instituto Carlos I de Física Teórica y Computacional, Universidad de Granada, Fuentenueva s/n, 18071 Granada, Spain*

We obtain a representation of pairing energies in phase space, for the Lipkin-Meshkov-Glick and general boson Bardeen-Cooper-Schrieffer pairing models. This is done by means of a probability distribution of the quantum state in phase space. In fact, we prove a correspondence between the points at which this probability distribution vanishes and the pairing energies. In principle, the vanishing of this probability distribution is experimentally accessible and additionally gives a method to visualize pairing energies across the model control parameter space. This result opens new ways to experimentally approach quantum pairing systems.

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Introduction. The concept of *pairing energies* is a fundamental ingredient in the Bardeen-Cooper-Schrieffer (BCS) theory of superconductivity [1]. The BCS Hamiltonian eigenvalues can be written as a sum of the energies of quasiparticles which are the called pairing energies. The BCS theory has also been widely applied to describe pairing correlation between nucleons in finite nuclei [2]. Here, we will consider two pairing models: the Lipkin-Meshkov-Glick (LMG) and a more general bosonic BCS-like model. The LMG Hamiltonian [3] is a nuclear mean field plus state dependent pairing interaction model, used to describe the quantum phase transition from spherical to deformed shapes in nuclei, which can be solved analytically [4]. The LMG model has been also employed in other fields of physics to describe many body systems such as quantum optics (to generate spin squeezed states and to describe many particle entangled states) [5] or in condensed matter physics (to characterize Bose-Einstein condensates and Josephson junctions) [6]. The bosonic BCS-like model is also an exactly solvable pairing model that has been proposed to model quantum phase transitions to a fragmented state for repulsive pairing interactions [7–9]. Boson models of arbitrary angular momentum involving repulsive pairing interactions have been used to support the validity of the interacting boson model in nuclear structure [10].

On the other hand, we can represent a quantum state ψ in phase space by means of a quasiprobability distribution used in quantum mechanics, the so-called Husimi distribution Q_ψ of a quantum state ψ (called Q -function in quantum optics), which can be defined as the squared overlap between ψ and an arbitrary coherent state, (a rigorous definition will be given below) and which plays a fundamental role in many branches of quantum physics, mainly in the study of quantum optics in phase space. Here we shall focus on the zeros of this phase-space representation of ψ , their physical meaning and their relation to pairing energies in the pairing model. It is known that the zeros of the phase-space distribution Q_ψ determine the quantum state ψ [11]. The zeros of Q_ψ are experimentally measurable for some cases, which provides

a way to reconstruct the associated quantum state ψ , a basic problem in quantum information theory. For example, for spin systems, the vanishing of the Q_ψ at a point in phase space can be tested in principle, by means of a Stern-Gerlach apparatus [see later for a discussion on experimental setups]. Also, the time evolution of coherent states of light in a Kerr medium is visualized by measuring Q_ψ by cavity state tomography, observing quantum collapses and revivals and confirming the non-classical properties of the transient states [12]. Moreover, the zeros of this phase-space probability have been shown as an indicator of the regular or chaotic behavior in quantum maps, theoretically, in a variety of quantum problems such as molecular systems [13], atomic physics [14], quantum Billiards [15] or in condensed matter physics [16] (see also [11, 17, 18] and references therein), and experimentally [19] in the kicked top. They have also been considered as an indicator of metal-insulator phase transitions [20] and quantum phase transitions in the Dicke and vibron models [21]. Additionally, in Refs. [22, 23], the eigenfunctions of the LMG Hamiltonian were analyzed in terms of the zeros of the Majorana polynomial (proportional to the Husimi amplitude), leading to exact expressions for the density of states in the thermodynamic, mean field, limit. All these examples support the important operational and physical meaning of the Q_ψ zeros and its experimental accessibility.

In this work, we establish a rigorous correspondence between the zeros of the phase-space probability distribution Q_ψ and the pairing energies in the LMG model. As a byproduct, a reconstruction of the wave function in terms of the zeros of Q_ψ is also provided. Besides, an analysis of the pairing energies' degeneracy is done in terms of the zeros' multiplicity. Finally, we show that the link between pairing energies and zeros of the phase-space probability distribution Q_ψ can be generalized to other pairing models like, for example, the bosonic BCS-like model.

Pairing Hamiltonians and the LMG model. Many models describing pairing correlations in condensed matter and nuclear physics are defined by a BCS-like Hamiltonian of

the form

$$H = \sum_{k,\sigma} \varepsilon_k^\sigma c_{k\sigma}^\dagger c_{k\sigma} + \sum_{kl,\sigma\tau} \gamma_{kk'\tau\tau'}^\sigma c_{k\sigma}^\dagger c_{k'\sigma'}^\dagger c_{l\tau} c_{l'\tau'} \quad (1)$$

where $c_{k\sigma}^\dagger$ ($c_{k\sigma}$) creates (destroys) a fermion in the k state of the $\sigma = \pm$ level with energy ε_k^σ . The states $(k'\sigma')$ and $(l'\tau')$ are usually taken to be the conjugate (time-reversed) of $(k\sigma)$ and $(l\tau)$, but one can also let them independent. For some particular choices of ε_k^σ and couplings γ , the Hamiltonian (1) can be written in terms of the $su(2)$ quasispin collective operators

$$J_+ = \sum_k c_{k+}^\dagger c_{k-} = J_-^\dagger, \quad J_z = \frac{1}{2} \sum_{k\sigma} \sigma c_{k\sigma}^\dagger c_{k\sigma}. \quad (2)$$

This is the case of the LMG model, which assumes that the nucleus is a system of fermions which can occupy two levels $\sigma = \pm$ with the same degeneracy ($2j$), separated by an energy ε . In the quasispin formalism, the model Hamiltonian is [3]:

$$H = \varepsilon J_z + \frac{\lambda}{2} (J_+^2 + J_-^2) + \frac{\gamma}{2} (J_+ J_- + J_- J_+). \quad (3)$$

The λ term annihilates pairs of particles in one level and creates pairs in the other level and the γ term scatters one particle up while another is scattered down. The total angular momentum $\vec{J}^2 = j(j+1)$ and the total number of particles $N = 2j$ are conserved. This symmetry reduces the size of the largest matrix to be diagonalized from 2^N to $N+1$. For a Dicke state $|j, m\rangle$, the eigenvalue m of J_z gives the number $n = m+j$ of excited particle-hole pairs. H also commutes with the parity operator $\hat{P} = e^{i\pi(J_z+j)}$, so that temporal evolution does not connect states with different parity. It will be useful for us to make use of the boson (Schwinger) realization of the $su(2)$ operators in terms of two bosons a and b :

$$J_+ = b^\dagger a, \quad J_- = a^\dagger b, \quad J_z = \frac{1}{2} (b^\dagger b - a^\dagger a), \quad (4)$$

and the Dicke states $|j, m\rangle$ in terms of Fock states $|n_a = j-m, n_b = j+m\rangle$, with n_a and n_b the occupancy number of levels a and b .

Exact solvability of the LMG model. Making use of the previous Schwinger realization, the LMG model has been proved to be exactly solvable by mapping it to a $SU(1, 1)$ Richardson-Gaudin integrable model [9]. Introducing the new parameters $\gamma_x = \frac{2j-1}{\varepsilon}(\gamma + \lambda)$ and $\gamma_y = \frac{2j-1}{\varepsilon}(\gamma - \lambda)$ and using $t = \sqrt{|\gamma_x/\gamma_y|}$, the unnormalized eigenvectors of the LMG Hamiltonian are found to be

$$|\psi_{M,\nu}\rangle = \prod_{\alpha=1}^M \left(\frac{a^\dagger a^\dagger}{e_\alpha + t} + \frac{b^\dagger b^\dagger}{e_\alpha - t} \right) |\nu_a, \nu_b\rangle \quad (5)$$

where $e_\alpha \in \mathbb{C}$ are the so-called spectral parameters or *pairing energies*, and ν_a and ν_b are the seniorities. For integer j , the seniorities are equal to $\nu_a = \nu_b \equiv \nu = 0$

or 1, and the total number of pairs is $M = j - \nu$. The M pairing energies e_α can be determined by solving a coupled set of Richardson (nonlinear) equations [9, 24] and the eigenvalues of the LMG Hamiltonian are given in terms of pairing energies.

Phase-space probability distribution and correspondence between its zeros and pairing energies. Given a general state $|\psi\rangle = \sum_{n_a, n_b} c_{n_a, n_b} |n_a, n_b\rangle$, with $n_a + n_b = 2j$ the total occupancy number, the so-called Husimi distribution Q_ψ is defined as the squared modulus of the overlap

$$Q_\psi(\zeta) = |\langle \zeta | \psi \rangle|^2 \quad (6)$$

between $|\psi\rangle$ and an arbitrary spin- j coherent state

$$\begin{aligned} |\zeta\rangle &= \frac{1}{\sqrt{(2j)!}} \frac{(a^\dagger + \zeta b^\dagger)^{2j}}{(1 + |\zeta|^2)^j} |0, 0\rangle, \\ &= (1 + |\zeta|^2)^{-j} \sum_{m=-j}^j \binom{2j}{j+m}^{1/2} \zeta^{j+m} |j, m\rangle, \end{aligned} \quad (7)$$

where $\zeta = \tan(\theta/2)e^{-i\phi}$ is given in terms of the polar θ and azimuthal ϕ angles on the Riemann sphere. This phase-space probability distribution function is non-negative and normalized according to $\int_{\mathbb{S}^2} Q_\psi(\zeta) d\Omega(\zeta) = 1$ (for normalized $|\psi\rangle$), with integration measure (the solid angle) $d\Omega(\zeta) = \frac{2j+1}{4\pi} \sin\theta d\theta d\phi$.

Taking into account that $\langle 0 | a^n a^{\dagger m} | 0 \rangle = n! \delta_{nm}$ (and a similar equation for b), after a little bit of algebra we can calculate the Husimi amplitude of the eigenvector (5) in terms of the pairing energies e_α as

$$\langle \psi_{j-\nu, \nu} | \zeta \rangle = \frac{\sqrt{(2j)!}}{(1 + |\zeta|^2)^j} \zeta^\nu \prod_{\alpha=1}^{j-\nu} \left(\frac{1}{\bar{e}_\alpha + t} + \frac{\zeta^2}{\bar{e}_\alpha - t} \right). \quad (8)$$

$[\bar{e}_\alpha$ denotes complex conjugate]. Therefore, the zeros of this phase-space representation of the quantum state ψ can be analytically related with the pairing energies by the equation:

$$\zeta_\alpha^2 = \frac{t - \bar{e}_\alpha}{\bar{e}_\alpha + t}, \quad \alpha = 1, \dots, j - \nu \quad (9)$$

This is the main result of this letter.

As already said, it is known that one can represent each quantum state ψ by means of the zeros of its phase-space representation Q_ψ [11]. Here we have also found an explicit expression of the Hamiltonian eigenstates (5) in terms of the zeros of their phase-space representation Q_ψ when plugging $e_\alpha = t(1 - \bar{\zeta}_\alpha^2)/(1 + \bar{\zeta}_\alpha^2)$ in eq. (5).

For a general state $|\psi\rangle = \sum_{m=-j}^j c_m |j, m\rangle$, the Q_ψ amplitude has the form

$$\langle \psi | \zeta \rangle = \frac{\sum_{m=-j}^j \bar{c}_m \sqrt{\binom{2j}{j+m}} \zeta^{j+m}}{(1 + |\zeta|^2)^j}. \quad (10)$$

Finite zeros of Q_ψ are then calculated by solving the $2j$ -degree 'Majorana polynomial' (see e.g. [23] in this context) equation $\sum_{m=-j}^j \bar{c}_m \binom{2j}{j+m}^{1/2} \zeta^{j+m} = 0$. Besides,

we have an extra zero of $\langle\psi|\zeta\rangle$ at $\zeta = \infty$ ($\theta = \pi$: South pole) when $c_j = 0$ (in particular, when ψ belongs to the odd parity sector with $\nu = 1$). In total, the Q_ψ amplitude $\langle\psi|\zeta\rangle$ has $2j$ zeros counted with their multiplicity. Since $\pm\zeta_\alpha$ yield the same e_α for $\alpha = 1, \dots, j - \nu$, then one always has $2j - 2\nu$ zeros attached to $j - \nu$ pairing energies. For $\nu = 1$, the extra 2ν zeros are $\zeta = 0$ and $\zeta = \infty$, as can be seen from eq. (8). All these values determine the eigenfunction (5) of the Hamiltonian. This can be done for any eigenstate of the Hamiltonian, that is finding the corresponding zeros of the Husimi function.

In particular, as an example, we have computed numerically the coefficients c_m of the ground state of the LMG Hamiltonian (3), and the corresponding zeros of Q_ψ , for the trajectory $\gamma_x = -\gamma_y + 10$ in the control parameter space. Figure (1) shows pairing energies e_α (first and second panels) and zeros $\zeta_\alpha = \tan(\theta_\alpha/2)e^{-i\phi_\alpha}$ of the Q_ψ function (third and fourth panels) for the ground state and the trajectory $\gamma_x = -\gamma_y + 10$. We have plotted the range $0 < \gamma_x < 10$. This Figure shows a clear example of the mapping between the pairing energies e_α and the zeros ζ_α of Q_ψ .

Multiplicity of zeros and degeneracy of pairing energies. Additionally one can appreciate the presence of collapses (vertical grey lines in Fig. 1) for certain values of the parameter γ_x of the LMG Hamiltonian. These collapses or places where several zeros of the Husimi function or pairing energies join are determined by the expression

$$\gamma_x = 5 \pm \sqrt{5^2 - \left(\frac{2j-1}{2j-1-2k}\right)^2}, k = 0, \dots, j-1, \quad (11)$$

for the trajectory $\gamma_x = -\gamma_y + 10$. This expression is obtained at the intersections between the straightline $\gamma_x = -\gamma_y + 10$ and the family of hyperbolas $\gamma_x\gamma_y = [(2j-1)/(2j-1-2k)]^2$ with $k = 0, \dots, j-1$. Notice that the point γ_x given in (11) must be real and then it exists if $[k \leq 2/5(2j-1)]$. The symbol $[x]$ means the smallest integer part of x . This family of points $(\gamma_x, \gamma_y = 5)$ has the property that each element verifies that the ground state amplitude of the Husimi function, $\langle\zeta|\psi\rangle$, has $j-k$ different zeros, one of them of multiplicity $2(k+1)$ and the others $j-k-1$ of multiplicity 2. Therefore when the path $\gamma_x = -\gamma_y + 10$, crosses each point of the above family, the zeros of the ground state Q_ψ function join with the aforementioned multiplicity. We want to stress that due to the relation given in Eq. (9) the same behavior is displayed by the pairing energies. For the trajectory $\gamma_x = \gamma_y$ with $\gamma_x > -1$, one verifies that the ground state Husimi function has only one zero (the north pole in the Riemann sphere) of multiplicity $2j$, so that all the zeros of the Q_ψ function (and thus all the pairing energies) join at the point $\gamma_x = \gamma_y = 5$. The coefficients appearing in the family of hyperbolas, that is $-(2j-1)/(2j-1-2k)$, define the γ_x values where crossings between the even and odd energy levels when one takes $\gamma_x = \gamma_y$ (these points are explicitly calculated in [31]).

Experimental setups for state reconstruction and the determination of zeros and pairing energies. Concerning

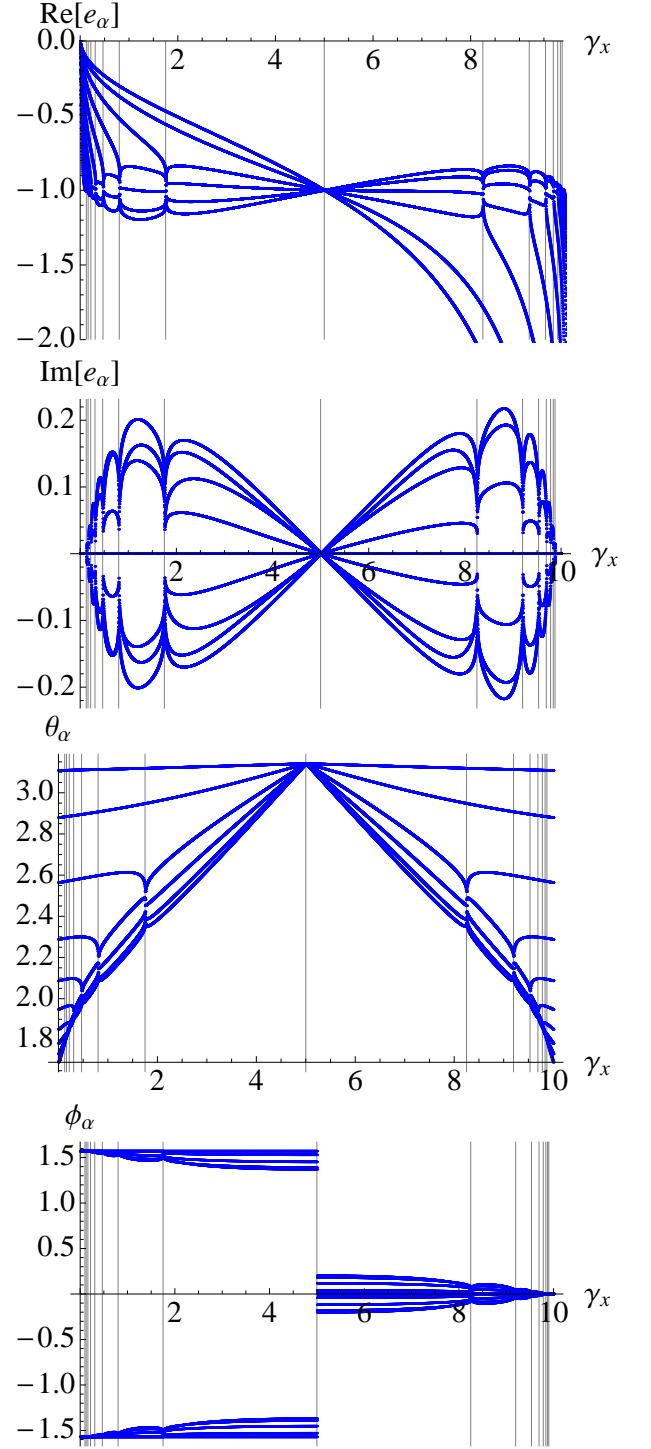


FIG. 1: (Color online) Ground state real and imaginary parts (first and second panels) of pairing energies and polar and azimuthal angles of zeros (third and fourth panels) of the phase-space probability distribution Q_ψ representing the ground state, for $j = 10$, associated to the trajectory $\gamma_x = -\gamma_y + 10$ for $0 < \gamma_x < 10$. The values of γ_x where pairing energies or zeros degenerate are indicated by vertical lines.

the physical meaning and the experimental determination of the zeros of Q_ψ for a spin system like LMG, let us discuss two possible experimental setups. The first one is related to the fact that the coherent state $|\zeta\rangle$ is the rotation of $|j, -j\rangle$ about the axis $\vec{r} = (\sin \phi, -\cos \phi, 0)$ in the x - y plane by an angle θ , that is $|\zeta\rangle = \exp(-i\theta\vec{r}\cdot\vec{J})|j, -j\rangle$. A possible procedure for quantum-state reconstruction is explained in References [25, 26]. Basically, the phase-space distribution $Q_\psi(\zeta) = |\langle\zeta|\psi\rangle|^2$ is precisely the probability to measure $m = -j$ in ψ , which can be determined by means of, for example, a Stern-Gerlach apparatus oriented along $\vec{R}(\theta, \phi) = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$ (see e.g. Ref. [25]). Repetition of this procedure (with a large number of identically prepared systems) for a finite number of orientations (θ, ϕ) , one can determine the function Q_ψ and, therefore, its zeros. Actually, the zeros ζ_0 of Q_ψ are just those orientations $\vec{R}(\theta_0, \phi_0) \leftrightarrow \zeta(\theta_0, \phi_0)$ of the Stern-Gerlach apparatus for which the probability of the outcome $m = -j$ vanishes. Simulations of the LMG model in a Bose-Einstein condensate in a double-well potential are known (see e.g. [27]). In this context, the polar angle θ is related to the population imbalance $j \cos \theta$ (the mean value of J_z), and the azimuthal angle ϕ is the relative phase of the two spatially separated Bose-Einstein condensates. Both quantities can be determined in terms of matter wave interference experiments as it is shown in Refs. [28–30].

Therefore, the reconstruction of any state $|\psi\rangle$ and the possibility to measure pairing energies in terms of the zeros of Q_ψ is then experimentally accessible.

Extension to other bosonic BCS-like models. The correspondence between pairing energies and zeros of the probability distribution representing a state in phase space is also present in other higher-dimensional BCS models, although the relation is a little bit more subtle. Let us consider for example the bosonic counterpart of (1), where $c_{k\sigma}$ are now boson $b_{k\sigma}$ annihilation operators. We shall consider $L + 1$ scalar bosons b_0, b_1, \dots, b_L . In the case of uniform couplings $\gamma_{kkll'} = \gamma/4$, the complete set of eigenstates of this model is given by

$$|\psi_{M,\nu}\rangle = \prod_{\alpha=1}^M \left(\sum_{\ell=0}^L \frac{b_\ell^\dagger b_\ell^\dagger}{2\varepsilon_\ell - e_\alpha} \right) |\nu\rangle, \quad (12)$$

where $|\nu\rangle = |\nu_0, \dots, \nu_L\rangle$ is a state of $\nu = \sum_{\ell=0}^L \nu_\ell$ unpaired bosons and $\nu_\ell = 0, 1$ are the seniorities. The total number of particles is $N = 2M + \nu$, with M the number of paired bosons. As for the LMG model, each eigenstate $|\psi_{M,\nu}\rangle$ is completely determined by a set of M pairing

energies e_α (which are solutions of a set of coupled nonlinear Richardson's equations [9]) and their energies are given by $E_{M,\nu}(e) = \sum_{\ell=0}^L \varepsilon_\ell \nu_\ell + \sum_{\alpha=1}^M e_\alpha$. The Q_ψ probability distribution of any quantum state ψ for this model is the squared modulus of the overlap between $|\psi\rangle$ and a general $SU(L+1)$ coherent state

$$|\zeta\rangle = \frac{1}{\sqrt{N!}} \frac{(b_0^\dagger + \sum_{\ell=1}^L \zeta_\ell b_\ell^\dagger)^N}{(1 + \sum_{\ell=1}^L |\zeta_\ell|^2)^{N/2}} |0\rangle, \quad (13)$$

which is a generalization of (7) for $\zeta = (\zeta_1, \dots, \zeta_L) \in \mathbb{C}^L$. The Q_ψ amplitude of an eigenstate (12) turns out to be

$$\langle\psi_{M,\nu}|\zeta\rangle = \frac{\sqrt{N!} \prod_{\ell=1}^L \zeta_\ell^{\nu_\ell}}{(1 + \sum_{\ell=1}^L |\zeta_\ell|^2)^{N/2}} \times \prod_{\alpha=1}^M \left(\frac{1}{2\varepsilon_0 - \bar{e}_\alpha} + \sum_{\ell=1}^L \frac{\zeta_\ell^2}{2\varepsilon_\ell - \bar{e}_\alpha} \right). \quad (14)$$

Therefore, the zeros $\zeta_{\ell,\alpha}$, $\alpha = 1, \dots, M$ of $\langle\psi_{M,\nu}|\zeta\rangle$ lie now on L -dimensional complex ellipsoids

$$\frac{\zeta_{1,\alpha}^2}{\xi_{1,\alpha}^2} + \dots + \frac{\zeta_{L,\alpha}^2}{\xi_{L,\alpha}^2} = 1, \quad \xi_{\ell,\alpha}^2 = \frac{2\varepsilon_\ell - \bar{e}_\alpha}{\bar{e}_\alpha - 2\varepsilon_0}, \quad (15)$$

with semi-principal axes of complex “length” $\xi_{\ell,\alpha}$. Still, there is a correspondence between pairing energies and complex ellipsoids in phase space, where the Q_ψ distribution vanishes. Moreover, the occurrence (or absence) of $\zeta_\ell = 0$ as a zero of (14) means seniority $\nu_\ell = 1$ (or $\nu_\ell = 0$).

Conclusions. We have revealed an interesting relation between pairing energies and zeros of a phase space probability distribution representing the quantum state in pairing systems. Zeros are experimentally accessible and this gives a method to find pairing energies' multiplicities across the control parameter space. As a byproduct, knowing the zeros, one can reconstruct the corresponding state. These results are proven to be valid for a large class of pairing systems and, in principle, they could be extended to other systems where particles pairs emerge.

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